Can the fluctuations of a black hole be treated thermodynamically?

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Abstract

Since the temperature of a typical Schwarzschild black hole is very low, some doubts are raised about whether the fluctuations of the black hole can be treated thermodynamically. It is shown that this is not the case: the thermodynamic fluctuations of a black hole are considerably larger than the corresponding quantum fluctuations. Moreover the ratio of the mean square thermodynamic fluctuation to the corresponding quantum fluctuation can be interpreted as a number of the effective constituents of a black hole.

Black holes being exact vacuum solutions of Einstein equations of the gravitational field are, undoubtedly, geometric objects. On the other hand, due to quantum effects they acquire thermodynamic properties. Understanding the statistical origin of black hole thermodynamics is still a central problem in black hole physics. The thermodynamic quantities which describe a system in equilibrium are, almost always, very nearly equal to their mean values. But deviations from the mean values sometimes occur. They are called fluctuations. The existence of the fluctuations has important meaning because it demonstrates statistical nature of the thermodynamic laws. In the fluctuation theory a distinction is usually made between, strictly speaking, thermodynamic fluctuations and quantum ones. The conditions for the fluctuations to be thermodynamic are [1]

$$T \gg \hbar/\tau, \quad \tau \gg \hbar/T, \tag{1}$$

where T is the temperature of a system and τ is the characteristic time of change of a thermodynamic quantity. Otherwise, when T is too low or when τ is too small (a thermodynamic quantity fluctuates too rapidly) the fluctuations cannot be treated thermodynamically, and the purely quantum fluctuations dominate.

Typically the mass of a black hole is considerably larger than the Planck mass, $M \gg m_P$, and its temperature $T \sim m_P^2/M$ is very low. In addition, for a thermodynamic system τ is bounded by the time it takes for a perturbation to propagate through the system. That is, $\tau \geq R$, where R is the characteristic size of the system. For a black hole the characteristic size coincides with its gravitational radius $R_g = 2GM$ and $T \sim 1/R_g$. Thus for a black hole we expect [2]

$$T \sim \hbar/\pi\tau \tag{2}$$

and the violation of the conditions (1). In this connection, an important question arises: What is a relation between thermodynamic and quantum fluctuations of a black hole? In this note I want to compare quantum fluctuations of a Schwarzschild black hole with their thermodynamic counterparts.

We begin with thermodynamic fluctuations. As mentioned above, the black hole is a thermodynamical system with the temperature and entropy given by

$$T = \frac{1}{8\pi GM} = \frac{1}{4\pi R_g}, \tag{3}$$

$$S = \frac{\textit{Horizon area}}{4G} = \frac{\pi R_g^2}{l_P^2}\,, \tag{4}$$

which are connected by the first law of black hole thermodynamics in the form $dE = TdS$, where the internal energy E is identified with the black hole mass M. In thermodynamics the mean square fluctuations of the fundamental thermodynamic quantities are related to the specific heat C. For a black hole

$$C = T\left(\frac{\partial S}{\partial T}\right) = -\frac{1}{8\pi\, G\, T^2} = -2S. \tag{5}$$

As is well known, a negative heat capacity means that a system cannot be in stable equilibrium with an infinite reservoir of radiation at temperature T. A well known way to stabilize a black hole is to place it in a appropriate cavity

so that an environmental heat bath is finite [3]. In this paper we will assume that black hole has reached equilibrium with its own radiation, the whole system being enclosed in the appropriate cavity and we can use a canonical ensemble. Thus for the mean square fluctuations of the black hole internal energy (mass), inverse temperature, and entropy we have respectively:

$$\langle (\Delta M)^2 \rangle_{th} = \frac{\partial(-M)}{\partial \beta} = \frac{C}{\beta^2} \sim m_P^2, \tag{6}$$

$$\langle (\Delta \beta)^2 \rangle_{th} = -\frac{\partial(\beta)}{\partial M} = \frac{\beta^2}{C} \sim m_P^{-2}, \tag{7}$$

$$\langle (\Delta S)^2 \rangle_{th} = C = 2S, \tag{8}$$

where I have added the subscript 'th' to refer to the thermodynamic fluctuations and omitted signs connected with C. As is easily seen, (8) can be also rewritten in terms of the fluctuations in the particle numbers

$$\langle (\Delta N)^2 \rangle_{th} \sim N, \tag{9}$$

where

$$N = \frac{\textit{Horizon area}}{l_P^2} \tag{10}$$

can be interpreted as the number of independent constituents of a black hole. As is well known, such fluctuations in the particle numbers inheres in classical particles [1]. Since the mean square fluctuations (6) - (8) don't depend on any extensive black hole parameter, the meaning of the relative fluctuations is obviously empty.

Let us now consider quantum fluctuations. To make precise calculations we need a theory of quantum gravity. But it is still absent. Despite this we can make some order-of-magnitude estimates. Suppose, following Frolov and Novikov [3], that a fluctuation in the geometry occurs in a spacetime domain with a characteristic size R so that the value of the metric g deviates from the expectation value $\langle g \rangle$ by Δg. Then, since the curvature in the domain changes by a quantity of order $\Delta g / (R^2 \langle g \rangle)^2$, it follows that the change in the action I of the gravitational field is

$$\Delta I \sim \frac{\Delta g R^2}{\langle g \rangle G}. \tag{11}$$

The probability of such a quantum fluctuation is considerable if only $\Delta I \sim \hbar$ so that ([4], [5])

$$\frac{\Delta g}{\langle g \rangle} \sim \frac{l_P^2}{R^2}. \tag{12}$$

This relation being applied to a Schwarzschild black hole determines the fluctuation of the gravitational radius R_g ([3], [6], [7]),

$$\Delta R_g \sim \frac{l_P^2}{R_g}. \tag{13}$$

From (13) we immediately obtain the mean square fluctuations of the black hole mass

$$\langle (\Delta M)^2 \rangle_q \sim \frac{m_P^4}{M^2}, \tag{14}$$

inverse temperature $\beta = 1/T$,

$$\langle (\Delta \beta)^2 \rangle_q \sim \frac{1}{m_P^4 \beta^2}, \tag{15}$$

and entropy

$$\langle (\Delta S)^2 \rangle_q \sim 1, \tag{16}$$

where I have added the subscript 'q' to refer to the quantum fluctuations. As in the thermodynamic case, we can rewrite (16) in terms of the fluctuations in the particle numbers,

$$\langle (\Delta N)^2 \rangle_q \sim 1. \tag{17}$$

As is easily seen from (14)-(16), the relative quantum fluctuations, as opposed to the thermodynamic case, can have a meaning, and we can write

$$\frac{\sqrt{\langle (\Delta M)^2 \rangle_q}}{M} \sim \frac{1}{N}, \tag{18}$$

$$\frac{\sqrt{\langle (\Delta \beta)^2 \rangle_q}}{\beta} \sim \frac{1}{N}, \tag{19}$$

$$\frac{\sqrt{\langle (\Delta S)^2 \rangle_q}}{S} \sim \frac{1}{N}. \tag{20}$$

Consequently, for large N (which is true for a typical black hole) they are quite negligible.

Of course, quantum fluctuations of macroscopic observables should obey the uncertainty principle. Let the uncertainty in the value of the gravitational radius be equal (13); then the corresponding uncertainty in the value of the momentum is $\Delta p \sim M$. Taking into account (9), (10), we can represent it in the form

$$\Delta p \sim \Delta E \sim \Delta N m_P \sim N^{1/2} m_P \sim M. \tag{21}$$

This allows us to regard a black hole as a set of $N^{1/2}$ heavy Planckanian constituents with masses m_P. In other terms, since quantum fluctuations inhere in a system at $T \sim 0$, we can regard a black hole as a set of $N^{1/2}$ Planckanian oscillators in the ground state

$$M = \frac{1}{2G} R_g = \left(\frac{R_g}{l_P}\right) \frac{\hbar \omega_P}{2} = n \frac{\hbar \omega_P}{2}, \tag{22}$$

where $n = \left(\frac{R_g}{l_P}\right) \sim N^{1/2}$ and ω_P is the Planck frequency, $\omega_P \sim m_P$. Such an interpretation can have a close relation with the Sakharov's idea of induced gravity [8], [9]. As Sakharov suggested, the Einstein-Hilbert action can be induced in a theory with no gravitational action by integrating out the zero-point fluctuations of matter fields with Planck frequencies. As is well known, in a flat spacetime the number of normal modes of vibration per unit volume in the range of wave numbers from k to $k + dk$ are $\sim k^2 dk$. Since each mode of a vacuum oscillation has a zero-point energy, $\frac{1}{2}\hbar\omega = \frac{1}{2}\hbar ck$, it follows that the total density of zero-point energy of a matter field formally diverges as

$$\frac{\hbar}{2} \int_0^\infty k^3 dk. \tag{23}$$

As Sakharov pointed out, curving spacetime alters it. In a curved manifold the number of standing waves per unit frequency changes in such a way that the energy density of vacuum fluctuations becomes

$$\rho = A\hbar \int k^3 dk + B\hbar R \int k dk + \hbar [CR^2 + DR^{\alpha\beta} R_{\alpha\beta)}] \int k^{-1} dk \tag{24}$$

plus higher-order terms. Here R is the Riemann scalar curvature invariant, and the numerical coefficients $A, B, ...$ are of the order of magnitude of unity. By means of the renormalization procedure, the first term in (24) must be

dropped. The second term, according to Sakharov, is proportional to the Einstein-Hilbert action for the gravitational field if effective upper limit in formally divergent integral in the second term in (24) is taken of the order of magnitude of the reciprocal Planck length, $k_{cutoff} \sim \omega_P$. The higher order terms lead to corrections to Einstein's equations and are omitted. Let us now return to a black hole. Since for a black hole region $R = (R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})^{1/2} = (12R_g^2/r^6)^{1/2}$, it follows that the energy of vacuum fluctuations inside the radius R_g is

$$E \sim \rho R_g^3 = (\hbar R \int_0^{\omega_P} kdk) R_g^3 \sim \left(\frac{R_g}{l_P}\right)\frac{\hbar\omega_P}{2} = n\frac{\hbar\omega_P}{2} = \frac{1}{2G}R_g. \qquad (25)$$

This coincides with our interpretation of the black hole mass (22). In some sense this answers the question: Why is the black hole mass linear in the gravitational radius?

Let us now compare thermodynamic fluctuations with quantum ones. As is easily seen, the thermodynamic fluctuations of the main black hole quantities (6)-(8) are considerably larger than the corresponding quantum fluctuations (14)-(16). To be more precise, the mean square thermodynamic fluctuation equals N times the corresponding quantum fluctuation,

$$\langle(\Delta M)^2\rangle_{th} = N\langle(\Delta M)^2\rangle_q, \qquad (26)$$

$$\langle(\Delta \beta)^2\rangle_{th} = N\langle(\Delta \beta)^2\rangle_q, \qquad (27)$$

$$\langle(\Delta S)^2\rangle_{th} = N\langle(\Delta S)^2\rangle_q. \qquad (28)$$

This resembles a property of another thermodynamic systems (i.e., harmonic oscillators) if a mean square quantum fluctuation is taken as a corresponding thermodynamic fluctuation of one constituent. But at first sight this looks somewhat strange, because quantum fluctuations have non-thermal nature. Moreover, as is seen from (13), quantum fluctuations in geometry of a black hole are proportional not only to $\hbar$ but also to T (on the other hand, thermodynamic fluctuations of a black hole (6) don't depend on T). The fact is that a black hole possesses a very special property which singles it out; namely, its size and temperature are not independent parameters. In contrast, quantum fluctuations of an ordinary harmonic oscillator in its ground

state depend on its energy and don't depend on temperature. Under this condition fluctuations of a black hole have a dual nature. For example, we can rewrite the relation (13) as

$$G^{-1}\Delta R_g \sim T. \tag{29}$$

It exhibits the equipartition theorem with G^{-1} playing the role of the force, dM/dR_g. According to the theorem, when a system is in thermal equilibrium, each of its degrees of freedom contributes the amount $T/2$ to the total energy. Thus, in addition to (22), we have for the black hole mass one more form

$$M = N\frac{T}{2}\ , \tag{30}$$

and the specific heat is, therefore, independent of temperature and equal to $N/2$. But the formula (22) deals with the number n. Then why don't we use this number instead of N, that is, why not

$$M = nT? \tag{31}$$

Actually we can do it if we express (31) in terms of the local temperature. The point is that the equipartition theorem is valid only at high temperatures and therefore the formula (30) represents the black hole mass at high temperatures; on the contrary, (22) expressed in terms n represents the black hole mass at low temperatures. Thus if we want to use n and therefore the formula (31), we should write it as

$$M = nT_{loc}, \tag{32}$$

where the local temperature T_{loc} is related with the black hole temperature (3) by $T_{loc} = T/\chi$, and χ is the redshift factor, $\chi = (1 - R_g/r)^{1/2}$. We need high temperatures. Such temperatures are near the horizon. But at $r \to R_g$ the local temperature diverges. To get rid of the divergence we should restrict ourselves to some minimum physical distance from the horizon. If we take l_P as a cutoff so that $\chi = (1 - R_g/r)^{1/2} \approx l_P/2R_g$, we obtain

$$M = nT_{loc} = n\frac{T}{\chi} \approx \left(\frac{R_g}{l_P}\right)\frac{T}{\frac{l_P}{2R_g}} \approx \left(\frac{R_g}{l_P}\right)^2 T \approx N\frac{T}{2} \tag{33}$$

as required. Thus the redshift factor transforms the number of the effective constituents of a black hole from the value n at infinity (low temperature limit) to N at the horizon (high temperature limit).

What is left is to show that the equipartition theorem is valid. As is well known [1], it is valid if only the thermal energy T is considerably larger than the spacing between energy levels. Is this satisfied in our case? As is well known [1], the mean energy spacing of a system is given by

$$\delta E = \Delta E\, e^{-S}, \tag{34}$$

where S is the entropy of the system and the width ΔE is some energy interval characteristic of the limitation in our ability to specify absolutely precisely the energy of a macroscopic system. It is equal in order of magnitude to the mean fluctuation of energy of the system. This expression can be immediately applied to a black hole. Then, taking into account (4) and (6), we conclude that

$$T \gg \delta E. \tag{35}$$

Therefore the equipartition theorem remains valid for the black holes. Moreover the inequality (35) ensures that quantum fluctuations of a black hole (14)-(16) are negligible. Thus we have removed all doubts about the validity of the conditions (1) in the black hole case.

In this paper we have shown that despite a low temperature the fluctuations of a black hole can be treated thermodynamically. Moreover, it turns out that the mean square thermodynamic fluctuation of a black hole equals N times the corresponding quantum fluctuation, where N can be interpreted as a number of the effective constituents of a black hole. This can be an evidence of the complexity of a black hole.

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